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Static and non-static electrical solenoids

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Abstract. We propose the system of charge and current densities realizing static and non-static electric solenoids. There is a non-trivial electric vector potential outside the electrical cylindrical or toroidal solenoids. It cannot be eliminated by the gauge transformation. The non-static solenoids emit waves of electromagnetic potentials propagating with the velocity of light. Conditions under which they can be detected are discussed.

1. Introduction

Magnetic solenoids have a broad application in physics and technology (see the review of their properties in [1, 2]). Electrical solenoids (ES) are less well known objects. By these we mean the system of charge and currents generating an electrical field (EF) confined to a space region S of finite extension. Although EF disappears outside S the non-trivial scalar electric and magnetic vector potentials (VP) could be different from zero there. As far as we know, there are only few references treating this subject. In [3] an electrical toroidal solenoid (ETS) was constructed in terms of the non-physical current of magnetic monopoles. Non-static point-like and cylindrical ES were studied in [4, 5], respectively. In both of them non-vanishing charge and current densities were presented. In [6] an ES was suggested with non-vanishing current density (and in the absence of charge). It is the aim of the present paper to treat ES systematically. The plan of our exposition is as follows. In section 2, the main facts concerning the magnetic toroidal solenoid (MTS) with constant current in its winding are presented. In section 3, an alternative viewpoint on the MTS is given. It turns out that magnetic dipoles properly distributed inside the torus exactly reproduce VP of the MTS with constant current. The boundary conditions satisfied by the magnetic field (MF) strength and induction are discussed in section 4. We change the magnetic dipoles inside the torus for electric ones in section 5, thus obtaining ETS. Non-static solenoids are studied in section 6. They emit waves of electromagnetic potentials, propagating off the solenoid with the velocity of light. A possible scheme of experimental installation for their detection is discussed.

2. Some facts concerning MTS

Consider torus T

$$(\rho-d)^2+z^2=R^2. \tag{2.1}$$

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Introduce the coordinates $\tilde{R}$, ψ: $\rho = d + \tilde{R}\cos\psi$, $z = \tilde{R}\sin\psi$. The value $\tilde{R} = R$ corresponds to the surface of T. Let the constant poloidal current (figure 1) flow over its surface. The density of this current is

$$j = -\frac{gc}{4\pi}\frac{\delta(\tilde{R}-R)}{d+R\cos\psi}\,\boldsymbol{n}_\psi. \tag{2.2}$$

Here $g = 2NI/c$, N is the total number of turns in the poloidal coil, I is the current flowing in a particular turn, $\boldsymbol{n}_\psi$ is the unit vector, defining the current direction on the torus surface:

$$\boldsymbol{n}_\psi = \boldsymbol{n}_z \cos\psi - \boldsymbol{n}_\rho \sin\psi \qquad \boldsymbol{n}_\rho = \boldsymbol{n}_x \cos\varphi + \boldsymbol{n}_y \sin\varphi.$$

The constant g may also be expressed through the magnetic flux ϕ inside T: $g = \phi[2\pi(d-\sqrt{d^2-R^2})]^{-1}$. MF $\boldsymbol{B} = \boldsymbol{H} = 0$ outside T and $\boldsymbol{B} = \boldsymbol{H} = \boldsymbol{n}_\varphi(g/\rho)$ inside T. Here ρ is the distance of the particular point inside T from the torus symmetry axis: $\rho = d + \tilde{R}\cos\psi$. The VP of the MTS has been obtained in [7]; its properties were discussed in [8]. In the integral form, the non-vanishing cylindrical components of VP are:

$$A_z = \frac{g\sqrt{R}}{2\pi}\int_0^{2\pi} \mathrm{d}\varphi \frac{d-\rho\cos\varphi}{q^{3/2}} Q_{1/2}(\cosh\mu)$$

$$A_\rho = \frac{g\sqrt{R}}{2\pi}\int_0^{2\pi} \mathrm{d}\varphi \frac{\cos\varphi}{q^{3/2}} Q_{1/2}(\cosh\mu) \tag{2.3}$$

$$\cosh\mu = (r^2+d^2+R^2-2d\rho\cos\varphi)/2Rq \qquad q^2 = (\rho\cos\varphi - d)^2 + z^2.$$

$Q_v(x)$ is the Legendre function of the second kind. For the infinitely thin ($R \ll d$) TS these integrals can be taken in a closed form. Outside TS one has:

$$A_z = \frac{gR^2}{2(d\rho)^{3/2}}\frac{1}{\sinh\mu_1}[\rho Q^1_{1/2}(\cosh\mu_1) - dQ^1_{-1/2}(\cosh\mu_1)]$$

$$A_\rho = -\frac{gR^2}{2(d\rho)^{3/2}}\frac{z}{\sinh\mu_1} Q^1_{1/2}(\cosh\mu_1) \qquad \cosh\mu_1 = \frac{r^2+d^2}{2d\rho}. \tag{2.4}$$

At large distances this VP falls as r^{-3}

$$A_z \sim \tfrac{1}{8}\pi g\, \mathrm{d}R^2 \frac{1+3\cos 2\theta}{r^3} \qquad A_\rho \sim \tfrac{3}{8}\pi g\, \mathrm{d}R^2 \frac{\sin 2\theta}{r^3}.$$

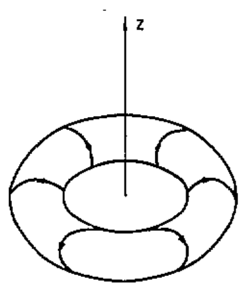

Figure 1. Poloidal current flowing on the torus surface.

3. An alternative viewpoint on the MTS

Instead of the poloidal current (2.2) one may equally well use the magnetization: $\boldsymbol{j} = c \operatorname{rot} \boldsymbol{M}$ [9, 10]. This is confined entirely inside the TS and given by

$$\boldsymbol{M} = M\boldsymbol{n}_\varphi \qquad M = \frac{g}{4\pi\rho}\,\theta(R - \sqrt{(\rho-d)^2 + z^2}). \tag{3.1}$$

Here $\theta(x)$ is the step function. For the infinitely thin TS, M reduces to

$$M = \frac{1}{4\pi}\,\phi\delta(\rho - d)\delta(z) \tag{3.2}$$

VP being expressed through the magnetization is

$$\boldsymbol{A}(\boldsymbol{r}) = \int \boldsymbol{M}(\boldsymbol{r}') \times \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3}\,\mathrm{d}V'. \tag{3.3}$$

What is the physical meaning of these relations? Equations (3.2) and (3.3) tell us that infinitely thin MTS can be realized as a closed chain of magnetic dipoles (figure 2). In fact, the value of VP at the point $\boldsymbol{r}$ produced by the elementary magnetic dipole at $\boldsymbol{r}'$ is given by [9]

$$\boldsymbol{A}(\boldsymbol{r}) = m\frac{\boldsymbol{n} \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3}. \tag{3.4}$$

Here $\boldsymbol{n}$ and m are the direction and power of the dipole, respectively. Integrating this equation over the circumference of the radius d lying in the $z = 0$ plane ($\boldsymbol{n} = \boldsymbol{n}_\varphi$, $\boldsymbol{r}' = d\boldsymbol{n}_\rho$) we arrive at (2.4) with $g = 4m/R^2$ or $\phi = 4\pi m/d$. Equations (3.1) and (3.3) mean that finite MTS may be realized as a closed spin tube of radius R. In fact, integrating (3.4) over the volume of T (by m in (3.4) one should understand the spin density coinciding with the magnetization (3.1)) we obtain (2.3). The closed spin tube (ferromagnetic ring with magnetization independent of applied fields) was used by Japanese physicists for experimental verification of the Aharonov–Bohm effect. The simpler case presents the cylindrical solenoid. It may be realized (figure 3) as a linear spin chain (or tube). In fact, integrating (3.4) over the Z axis we arrive at the VP of the cylindrical solenoid: $\boldsymbol{A} = \boldsymbol{n}_\varphi\phi/2\pi\rho$. Such a spin chain (magnetized whisker) was used in earlier experiments testing the AB effect (see their review in [12]).

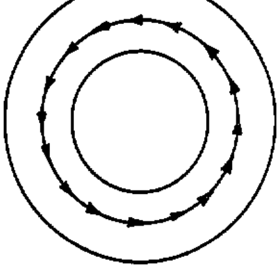

Figure 2. Explicit realization of the magnetic (electric) toroidal solenoid by means of the circular magnetic (electric) dipole chain.

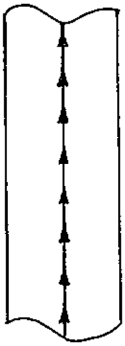

Figure 3. Explicit realization of the magnetic (electric) cylindrical solenoid by means of the linear magnetic (electric) dipole chain.

4. The magnetic field and boundary conditions

We write out the general conditions defining $\boldsymbol{B}$, $\boldsymbol{H}$, $\boldsymbol{M}$ [9]:

$$\operatorname{div}\boldsymbol{B}=0 \qquad \operatorname{rot}\boldsymbol{H}=\frac{4\pi}{c}\boldsymbol{j} \qquad \boldsymbol{B}=\boldsymbol{H}+4\pi\boldsymbol{M}. \tag{4.1}$$

For the TS with surface current, considered in section 2, $\boldsymbol{M}=0$; therefore $\boldsymbol{B}=\boldsymbol{H}$ both inside and outside the TS. On its boundary the normal component of $\boldsymbol{B}$ is continuous whereas the tangential component of $\boldsymbol{H}$ suffers a finite jump equal to the surface current density. On the other hand, for the magnetized spin tube treated in section 3 $\boldsymbol{j}=0$, $\boldsymbol{M}\neq 0$. It turns out [13], that for the static case the condition $\operatorname{div}\boldsymbol{M}=0$ guarantees that H vanishes everywhere while $\boldsymbol{B}$ differs from zero in those space regions, where $\boldsymbol{M}\neq 0$. Hence, it follows that the solenoid of an arbitrary geometrical form can be constructed by filling this form with the substance having the solenoidal magnetization. The typical example is the closed uniformly magnetized filament of an arbitrary form. Such filaments are used in experiments testing the existence of the AB effect [14]. Some care is needed when one uses the magnetization formalism. If, for the current in vacuum, we introduce (formally) fictitious magnetization ($\boldsymbol{j}=c\operatorname{rot}\boldsymbol{M}$) then $\boldsymbol{B}=\boldsymbol{H}$ everywhere. On the other hand, for the real medium with magnetization $\boldsymbol{M}$ we have (in the absence of current $\boldsymbol{j}$)

$$\boldsymbol{B}=\boldsymbol{H}+4\pi\boldsymbol{M}$$

$$\boldsymbol{H}=\operatorname{grad}\int G_0(\boldsymbol{r},\boldsymbol{r}')\operatorname{div}\boldsymbol{M}(\boldsymbol{r},\boldsymbol{r}')\,\mathrm{d}V' \qquad G_0=|\boldsymbol{r}-\boldsymbol{r}'|^{-1}.$$

Thus, $\boldsymbol{H}=0$ if $\operatorname{div}\boldsymbol{M}=0$. It follows from (4.1) that

$$\boldsymbol{B}=\operatorname{rot}\boldsymbol{A} \qquad \operatorname{div}\boldsymbol{A}=0 \qquad \Delta\boldsymbol{A}=-4\pi\boldsymbol{M}.$$

The solution of this last equation is just (3.3).

5. Electrical static solenoids

Now we replace the magnetic dipoles by electric ones. Then, all equations obtained in sections 3 and 4 remain the same. The electric polarization is given by $\boldsymbol{P}=P\boldsymbol{n}_\varphi$ where $P=\phi\delta(\rho-d)\delta(z)/4\pi$ for the circular dipole chain and $P=g\theta(R-\sqrt{(\rho-d)^2+z^2})/4\pi\rho$ for the finite ETS. Further, $g=\phi[2\pi(d-\sqrt{d^2-R^2})]^{-1}$ and

ϕ is the electric induction flux through the cross section of the solenoid. In the absence of free charges and external fields we have the following equations for $\boldsymbol{D}$, $\boldsymbol{E}$:

$$\operatorname{div} \boldsymbol{D} = 0 \qquad \operatorname{rot} \boldsymbol{E} = 0 \qquad \boldsymbol{D} = \boldsymbol{E} + 4\pi \boldsymbol{P}. \tag{5.1}$$

Eliminating $\boldsymbol{E}$ we obtain equations for $\boldsymbol{D}$:

$$\operatorname{div} \boldsymbol{D} = 0 \tag{5.2}$$

$$\operatorname{rot} \boldsymbol{D} = 4\pi \boldsymbol{P}. \tag{5.3}$$

To satisfy (5.2) we put $\boldsymbol{D} = \operatorname{rot} \boldsymbol{A}_l$, $\operatorname{div} \boldsymbol{A}_l = 0$ and obtain the following equation for $\boldsymbol{A}_e : \Delta \boldsymbol{A}_e = -4\pi \operatorname{rot} \boldsymbol{P}$. Its solution is

$$\boldsymbol{A}_e = \int \frac{\operatorname{rot} \boldsymbol{P}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{d} V' = \int \boldsymbol{P}(\boldsymbol{r}') \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \, \mathrm{d} V' \tag{5.4}$$

which coincides with (3.3). It follows from this that $\boldsymbol{D} = 4\pi \boldsymbol{P}$. Hence, $\boldsymbol{E} = 0$. On the other hand, we may exclude $\boldsymbol{D}$ from (5.1)

$$\operatorname{rot} \boldsymbol{E} = 0 \qquad \operatorname{div} \boldsymbol{E} = -4\pi \operatorname{div} \boldsymbol{P}. \tag{5.5}$$

To satisfy the first of these equations we put $\boldsymbol{E} = -\operatorname{grad} \phi$, thus arriving at

$$\Delta \phi = 4\pi \operatorname{div} \boldsymbol{P} \qquad \phi = -\int G_0(\boldsymbol{r}, \boldsymbol{r}') \operatorname{div} \boldsymbol{P}(\boldsymbol{r}') \, \mathrm{d} V'.$$

Since $\operatorname{div} \boldsymbol{P} = 0$ for the treated toroidal configuration, so $\phi = \boldsymbol{E} = 0$ everywhere and $\boldsymbol{D} = 4\pi \boldsymbol{P}$.

The appearance of VP is a somewhat unusual fact in electrostatic problems. To clarify its physical meaning we consider the prolate axially symmetric ellipsoid $\mathscr{E}$

$$\frac{\rho^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{5.6}$$

with constant polarization directed along the z axis. Introduce the spheroidal coordinates

$$\rho = a \sinh \mu \sin \theta \qquad z = a \cosh \mu \cos \theta.$$

Let the value $\mu = \mu_0$ correspond to the ellipsoid (5.6)

$$b = a \sinh \mu_0 \qquad c = a \cosh \mu_0. \tag{5.7}$$

Then, polarization is given by

$$\boldsymbol{P} = P_0 \boldsymbol{n}_z. \tag{5.8}$$

Equations (5.1) with polarization (5.8) may be solved in terms of scalar or vector electric potentials. In the first case one has:

$$\phi = 4\pi P_0 a \sinh^2 \mu_0 \cos \theta \, f_{10}(\mu, \mu_0)$$

$$\boldsymbol{E} = -\operatorname{grad} \phi = \frac{4\pi P_0 \sinh^2 \mu_0}{(\cosh^2 \mu - \cos^2 \theta)^{1/2}} \left[\sin \theta \, f_{10}(\mu, \mu_0) \boldsymbol{n}_\theta - \cos \theta \frac{df_{10}(\mu, \mu_0)}{d\mu} \boldsymbol{n}_\mu \right]. \tag{5.9}$$

Here:

$$f_{lm}(\mu, \mu_0) = \begin{cases} Q_l^m(\cosh\mu)P_l^m(\cosh\mu_0) & \text{for } \mu > \mu_0 \\ P_l^m(\cosh\mu)Q_m^l(\cosh\mu_0) & \text{for } \mu < \mu_0 \end{cases}$$

P_l^m and Q_l^m are the Legendre functions.

On the other hand, the same equations may be solved in terms of electric VP ($\boldsymbol{D} = \operatorname{rot} \boldsymbol{A}_e$). The answer is:

$$\boldsymbol{A}_e = A\boldsymbol{n}_\varphi \qquad A = -\pi P_0 a \sinh 2\mu_0 \sin\theta\, f_{11}(\mu, \mu_0). \tag{5.10}$$

Substituting $\boldsymbol{A}_e$ into (5.1) we arrive at $\boldsymbol{D}$, $\boldsymbol{E}$ obtained via the electric VP. Thus, there are two equivalent ways to find the solution for the permanently polarized ellipsoid. They are expressed through the electric scalar and vector potentials, respectively. As the space outside the ellipsoid is simply-connected, the potentials ϕ and $\boldsymbol{A}_l$ are uniquely defined by $\boldsymbol{D}$, $\boldsymbol{E}$ and, thus, they both have only auxiliary meaning. Now let the major semiaxis (c) of the ellipsoid (5.6) tend to infinity while the minor one (b) remains the same. For this it is enough to put $a = b/\sinh\mu_0$ in (5.9) and (5.10) and then take the limit $\mu_0 \to 0$. It turns out that in this limit $\phi \to 0$, $\boldsymbol{E} \to 0$ everywhere. Further, $\boldsymbol{D} \to 0$, $A \to 2\pi P_0 b^2/\rho$ outside the ellipsoid and $\boldsymbol{D} \to 4\pi \boldsymbol{P}$, $A_e \to 2\pi P_0 \rho$ inside it. Thus, we recover the VP of the electric cylindrical solenoid (into which the ellipsoid $\mathscr{E}$ degenerates). The same situation (that is the disappearance of $\boldsymbol{E}$, ϕ and the surviving of $\boldsymbol{A}_e$, $\boldsymbol{D}$) takes place for the toroidal solenoid. The moral of these considerations is that in general the electric VP has equal status with the scalar one. There are special situations in electrostatics in which either electric scalar or vector potentials survive. The electric toroidal and cylindrical solenoids are of the latter kind.

One may wonder why we limit ourselves to the consideration of such complicated objects as toroidal solenoids? The reason is that non-trivial (ones which cannot be removed by the gauge transformation and which, thus, have a chance to be observed experimentally) vector potentials (magnetic or electric) may exist only if the space regions where $E = H = 0$ are multiconnected ones. The exterior of the cylinder or torus are the simplest examples of such spaces. However, the finite length of the real cylindrical solenoid leads to the appearance of return magnetic (or electric) flux outside the solenoid. This in turn complicates the unambiguous interpretation of the AB type experiments. Thus, toroidal configuration seems to be most suitable for our purposes.

To the best of our knowledge there are only a few references in which the electric VP is studied. A short remark on these potentials may be found in the Stratton treatise [15] and in [16]. In 1990 Ventura [17] introduced an alternative representation of E in terms of electric VP and used it for the quantization of the electron Coulomb field. In an interesting paper [18] Jefimenko used electric VP as an intermediate step for the simplified derivation of the usual Lorentz force. Finally, it was suggested in [19] that electric VP could be generated by the pseudovector Dirac current $\psi^+\gamma_5\alpha\psi$. In all these references the presentation of $\boldsymbol{D}$ in the form $D = \operatorname{rot} \boldsymbol{A}_e$ was considered to be equivalent to the usual one $\boldsymbol{E} = -\operatorname{grad}\phi$. It is the aim of this paper to present concrete physical situations for which the electric vector potential, not the scalar one, has physical meaning.

How to verify the existence of the electric field inside the ETS? We use the same means as for the MTS. We briefly enumerate them:

(1) the electromagnetic field appears outside the solenoid when it moves uniformly in the medium with $\varepsilon\mu \neq 1$ [20];

(2) the electromagnetic field appears outside the accelerated solenoid (both in vacuum and medium) [1];

(3) the interaction of the external electric field (EF) with the electric dipoles confined inside the ETS is given by

$$U=-\int \boldsymbol{E}_{\mathrm{ext}}\boldsymbol{P}\,\mathrm{d}V. \tag{5.11}$$

At large distances from the source of external EF (or for small dimensions of the ETS) this reduces to

$$U=-\tfrac{1}{2}\boldsymbol{\varepsilon}_t \operatorname{rot}\boldsymbol{E}_{\mathrm{ext}}=\frac{1}{2c}\,\boldsymbol{\varepsilon}_t\dot{\boldsymbol{H}}_{\mathrm{ext}}. \tag{5.12}$$

Here $\boldsymbol{\varepsilon}_t$ is the so-called toroidal electric moment [3]. For the polarization P given above $\boldsymbol{\varepsilon}_t$ is directed along the torus symmetry axis and is equal to $\frac{1}{2}\pi g d R^2$. Equation (5.12) means that at large distances the ETS interacts with the time varying MF. This equation was used in [21] to explain the observed rotation [22] of non-magnetic molecules in the uniform MF slowly varying with time.

In the examples considered so far we have either forced the electromagnetic field to come out of the ETS by putting it into motion, or permitted the external EF to penetrate inside the ETS and interact with electrical dipoles confined there. There is a non-vanishing electric VP outside the ETS which cannot be eliminated by the gauge transformation as $\oint A_l\,\mathrm{d}l=\phi$ for the closed contours passing through the torus hole. Can we prove the existence of the electric VP by making observations outside the ETS? (a suitable screen can be used to prevent the penetration of incoming charged particles into the ETS). We do not have the obvious answer. In fact, the analogue of the AB effect for this case would be quantum scattering of free magnetic charges by the electric VP. However, these particles (monopoles) have not been found in nature up to now.

It is rather curious that superposing the electric and magnetic dipole distributions inside the torus we get the electromagnetic toroidal solenoid. The electromagnetic inductions differ from zero only inside the torus. Outside it there are non-vanishing electric and magnetic vector potentials.

The question arises as to the technical realization of the ETS. There exist neutral dielectrics called electrets that carry non-zero static electric dipole moments [23]. Among different types of electrets, the most suitable seems to be the ferro-electrics which are the electric analogues of ferromagnetics. From these substances the electrified ring can be manufactured, in exactly the same way as the magnetized ring in Tonomura's experiments [11, 14].

6. Non-static electric solenoids

Consider charge and current densities periodically changing with time. In what follows we shall frequently omit the common factor $\exp(-\mathrm{i}\omega t)$. It should be restored when the time differentiation is performed or in final expressions from which the real part should be taken. In order for the continuity equation $\operatorname{div}\boldsymbol{j}+\dot{\rho}=0$ to be satisfied automatically, we choose ρ and $\boldsymbol{j}$ in the form

$$\rho=\exp(-\mathrm{i}\omega t)\Delta f \qquad \boldsymbol{j}=\mathrm{i}\omega\exp(-\mathrm{i}\omega t)\nabla f. \tag{6.1}$$

The corresponding electromagnetic potentials and field strengths are:

$$\phi=-4\pi f-k^2\int G_k(\boldsymbol{r},\boldsymbol{r}')f(\boldsymbol{r}')\,\mathrm{d}V' \qquad \boldsymbol{E}=4\pi\nabla f$$
$$\boldsymbol{A}=\mathrm{i}k\nabla\int G_k(\boldsymbol{r},\boldsymbol{r}')f(\boldsymbol{r}')\,\mathrm{d}V' \qquad \boldsymbol{H}=0 \tag{6.2}$$
$$G_k=\exp(\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|)/|\boldsymbol{r}-\boldsymbol{r}'| \qquad \operatorname{div}\boldsymbol{A}+\frac{1}{c}\dot{\varphi}=0.$$

It follows from this that for the function f being confined to the finite region of space S the same is valid for the electric strength E. On the other hand, the electromagnetic potentials are different from zero both inside and outside S. Thus, densities (6.1) realize the non-static electric solenoid. In particular, the function f may be taken to be non-vanishing inside the torus $(\rho-d)^2+z^2=R^2$ only. For this, one may simply put $f=D\theta(R-\sqrt{(\rho-d)^2+z^2})$.

The point-like realization [4] of the electric solenoid is obtained when the δ function choice of function f is made $\rho=D\Delta\delta^3(\boldsymbol{r})$, $\boldsymbol{j}=\mathrm{i}\omega D\nabla\cdot\delta^3(\boldsymbol{r})$

$$\phi=-D[4\pi\delta^3(\boldsymbol{r})+k^2\exp(\mathrm{i}kr)/r]$$
$$\boldsymbol{A}=\mathrm{i}kD\nabla\exp(\mathrm{i}kr)/r \qquad \boldsymbol{E}=4\pi D\nabla\delta^3(\boldsymbol{r}) \qquad \boldsymbol{H}=0. \tag{6.3}$$

The realization of the cylindrical electric solenoid via the cylindrical capacitor was proposed in [5]. Beautiful experiments with it were described in [24]. More interesting is the case of the spherical capacitor which is obtained when the following choice of ρ and $\boldsymbol{j}$ is made

$$\rho=\frac{e}{4\pi r^2}[\delta(r-r_1)-\delta(r-r_2)] \qquad \boldsymbol{j}=\mathrm{i}\omega e\frac{\boldsymbol{r}}{r^3}\theta(r-r_1)\theta(r_2-r) \tag{6.4}$$

This means that the charge density differs from zero only on the spherical shells ($r=r_1$ and $r=r_2$) where it periodically changes with time (take into account omitted factor $\exp(-\mathrm{i}\omega t)$). The periodical current $\boldsymbol{j}$ flows between these shells in the radial direction. The corresponding scalar and vector potentials (only the radial component of the latter differs from zero) may be found in [25]. It turns out that the magnetic field equals zero everywhere, while the electric one differs from zero only inside the capacitor ($r_1<r<r_2$): $\boldsymbol{E}=e\boldsymbol{r}/r^3$.

It follows from (6.2), (6.3) and [25] that outside the non-static electric solenoids there exist waves of electromagnetic potentials (EP waves for short). They propagate off the source with the velocity of light. As $E=H=0$ inside them, they do not carry electromagnetic energy. This means that they can be observable (if ever) on the quantum level only (as the electromagnetic potentials enter into the Schrödinger equation). Let the space region S, where $E\neq 0$, be inaccessible for the observer (a suitable screen can be used). Can he prove the existence of EP waves operating entirely outside S? At first it seems to be impossible. In fact, the transformation

$$\boldsymbol{A}\to\boldsymbol{A}'=\boldsymbol{A}-\nabla\chi \qquad \phi\to\phi'=\phi+\frac{1}{c}\dot{\chi}$$
$$\psi\to\psi'=\psi\exp(-\mathrm{i}e\chi/\hbar c) \qquad \chi=\mathrm{i}k\int G_k(\boldsymbol{r},\boldsymbol{r}')f(\boldsymbol{r}')\,\mathrm{d}V'$$

removes electromagnetic potentials outside S. If the function χ is single-valued (continuous) outside S, then ψ and ψ' describe the same physical situation. In all cases considered in this section the electromagnetic potentials can be removed without spoiling the singlevaluedness properties of the wavefunction and, thus, they are not observable. In the static limit (6.1) and (6.2) become trivial: $\rho=\Delta f$, $j=0$, $\phi=4\pi f$, $A=0$, $E=4\pi\nabla f$, $H=0$. At the present moment we are unable to construct non-static solenoids having, in the static limit, the non-trivial electric solenoids considered in section 5.

7. Non-static electromagnetic solenoids

Two conditions should be fulfilled for the observability of EP waves. First, the space surrounding the region S, where E, $H\neq 0$ should be multiconnected. Second, the function χ eliminating the electromagnetic potential outside S should be multivalued. We propose the following experimental arrangement for the generation and detection of EP waves (figure 4). The beam of charged particles (e.g. electrons) is scattered by the impenetrable torus inside which there are non-radiating charge and current densities periodically changing with time. The phase of the wavefunction is modulated for the electrons passing through the torus hole. As a result an interference picture periodically changing with time arises at the screen which may be sufficiently separated from the torus. Thus, there appears a principal possibility to transfer the information without transferring the electromagnetic energy. Certainly, the energy is needed to sustain the current inside the torus (by the way, it may be superconducting) as well as incoming electron beam. The main problem is to create charge and current densities generating non-trivial EP waves. The following particular example illustrates the difficulties. Let ρ and j be of the form

$$\rho=\mathrm{i}k(\operatorname{div}\boldsymbol{N}+\Delta\psi) \qquad \boldsymbol{j}=(\operatorname{rot}\operatorname{rot}\boldsymbol{N}-k^2\boldsymbol{N})c-ck^2\nabla\psi. \tag{7.1}$$

Here ψ and $\boldsymbol{N}$ differ from zero only inside torus. The corresponding potentials and field strengths are

$$\begin{aligned}&\phi=\mathrm{i}k\chi-4\pi\mathrm{i}k\psi \qquad \boldsymbol{A}=\nabla\chi+4\pi\boldsymbol{N} \qquad \chi=\int G(\operatorname{div}\boldsymbol{N}-k^2\psi)\,\mathrm{d}V'\\&\boldsymbol{H}=4\pi\operatorname{rot}\boldsymbol{N} \qquad \boldsymbol{E}=4\pi\mathrm{i}k(\boldsymbol{N}+\nabla\psi).\end{aligned} \tag{7.2}$$

Again there are EP waves outside the TS. If ϕ is the magnetic flux inside the TS and C is the closed contour lying outside the TS and passing through its hole, then

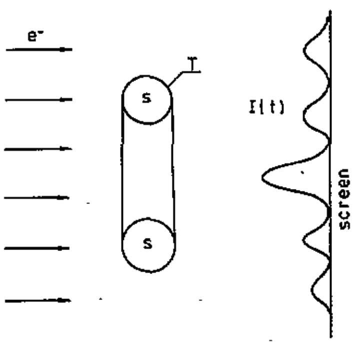

Figure 4. The schematic presentation of installation for generation and registration of EP waves.

$\int \boldsymbol{H}\,\mathrm{d}\boldsymbol{S}=\oint A_l\,\mathrm{d}l=\phi$. Since $\boldsymbol{N}=0$ outside the TS so χ is a multivalued function there. Further, the function $\boldsymbol{N}$ should not be well-behaved. This may be seen from the Stokes theorem according to which $\int \boldsymbol{H}\,\mathrm{d}\boldsymbol{S}=4\pi\int \operatorname{rot}\boldsymbol{N}\,\mathrm{d}\boldsymbol{S}=\phi=4\pi\int N_l\,\mathrm{d}l$. This means that $\boldsymbol{N}\neq 0$ outside the TS, which contradicts our earlier assumption on the vanishing of $\boldsymbol{N}$ outside the TS. But the Stokes theorem is valid only for those functions which are not too singular. So, $\boldsymbol{N}$ and, as a consequence, $\boldsymbol{j}$ and $\boldsymbol{E}$ are singular inside the TS. By the way, the singularity of $\boldsymbol{E}$ invalidates the fact that $\boldsymbol{E}\neq 0$ outside the TS (as one may erroneously deduce by applying the Stokes theorem to both sides of the Maxwell equation $\operatorname{rot}\boldsymbol{E}=-\dot{\boldsymbol{H}}/c$).

It is not clear how to solve the Helmholtz equation with such a singular current. Further, we cannot use the multipole expansion of the wave Green function G in terms of the usual spherical harmonics as we lose the non-singlevaluedness property of the function χ (the contour passing through the TS hole cannot be parametrized in terms of polar angle θ). On the other hand, the expansion of the wave Green function does not exist in terms of the toroidal coordinates (which are suitable for the description of the above contour). Up to now, we have not succeeded in obtaining an explicit expression for $\boldsymbol{N}$ which meets the above demands. This makes questionable the particular realization (7.1). However, the situation with static TS gives a hint that these complications are of a technical nature, not problems of principle. In fact, there are known VP of static TS [7] and a multivalued function χ satisfying $\boldsymbol{A}=\nabla\chi$ [26]. It is also possible to find the singular function $\boldsymbol{N}$ ($\boldsymbol{H}=4\pi\boldsymbol{N}$) vanishing outside the TS (see [27] and references therein). The use of the $\boldsymbol{N}$ function is justified for static TS and we do not see obvious reasons why it should not work for the non-static case. This is confirmed by recent considerations [28] of the AB effect in terms of time-dependent VP. It should be noted that the idea of the existence of EP waves is not new [5, 29, 30]. The key problem, however, is to find physical conditions under which these waves become observable (similarly to the observation [11] of static VP in the usual AB effect).

Up to now we have considered non-static solenoids in which both charge and current densities were different from zero. Is it possible to construct a non-static solenoid by using only the time-dependent current density? The following current configuration was proposed in [6]. The torus $T((\rho-d)^2+z^2=R^2$, see figure 5) is densely covered by the infinitely thin toroidal solenoids ts_i. It is claimed in [6] that for the periodical current in the windings of ts_i the electromagnetic strengths are confined to the interior of T. We prove now that this does not take place for the infinitely thin torus T. The set of toroidal solenoids ts_i (being infinitely small now) can be viewed as a circular chain of toroidal moments (TM) [31]:

$$\boldsymbol{t}=\exp(-\mathrm{i}\omega t)t\boldsymbol{n}_\varphi\delta(\rho-d)\delta(z). \qquad (7.3)$$

(See figure 2 where arrows now mean $\boldsymbol{t}$.) To this TM there corresponds the current $\boldsymbol{j}$

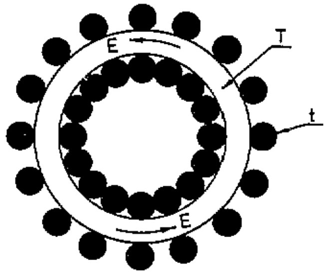

Figure 5. The torus T is 'dressed' by the toroidal solenoids ts_i with alternate currents in their coils. For very thin solenoids ts_i and dense covering of T by the electromagnetic strengths are confined inside T (according to [6]).

given by $j = c\,\mathrm{rot}\,\mathrm{rot}\,t$ [31]. The VP generated by this chain of TM is $A = \int G_k(r, r')\,\mathrm{rot}\,\mathrm{rot}\,t(r')\,\mathrm{d}V'$ (the factor $\exp(-\mathrm{i}\omega t)$ is again omitted). Here integration is performed along the circumference of the radius d. Integrating twice by parts we get

$$A = \mathrm{rot}\,\mathrm{rot}\int G_k t\,\mathrm{d}V' = (\mathrm{grad}\,\mathrm{div} + k^2)n_\varphi I + 4\pi t(r).$$

Here

$$I = \int_0^{2\pi} \frac{\exp(\mathrm{i}kZ)}{Z}\cos\varphi\,\mathrm{d}\varphi \qquad Z = (r^2 + d^2 - 2d\rho\cos\varphi)^{1/2}.$$

As I is independent of φ, so $\mathrm{div}(In_\varphi) = 0$ and

$$A = 4\pi t(r) + k^2 I n_\varphi. \tag{7.4}$$

The exact value of I was found in [25]. It is given by

$$I = \frac{\mathrm{i}\pi^2}{(\tilde{d}\tilde{r})^{1/2}} \sum_{n=0}^{\infty} (2n + \tfrac{3}{2}) \frac{1}{2^{4n}} \frac{2n+1}{n+1} \binom{2n}{n}^2 J_{2n+3/2}(k\tilde{d}) H^{(1)}_{2n+3/2}(k\tilde{r}).$$

Here

$$\tilde{r} = \tfrac{1}{2}(r_1 + r_2) \qquad \tilde{d} = \tfrac{1}{2}(r_1 - r_2) \qquad r_{1,2} = [(\rho \pm d)^2 + z^2]^{1/2}.$$

Hence, I is certainly different from zero. Then outside the chain of TM there are non-vanishing field strengths:

$$E_\varphi = \mathrm{i}k^3 I \qquad H_r = \frac{k^2}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, I) \qquad H_\theta = -\frac{k^2}{r}\frac{\partial}{\partial r}(rI)$$

Thus, E and H go beyond the interior of T and this completes our proof. It is valid for the infinitely thin torus T. But the measurements in [6] were performed for a torus of finite radius. It is known that for the cylindrical solenoid with periodical current in its winding the vector potential disappears outside the solenoid for specific values of kR ($k = \omega/c$, R is the radius of the solenoid). Suppose that the same property takes place for the current configuration of figure 5. Then, the disappearance of the electromagnetic strengths outside T observed in [6] may be attributed to the proximity of the torus radius to the specific value mentioned above.

In sections 3–5, we have constructed static electric and magnetic solenoids using closed chains of electric and magnetic dipoles, respectively. What happens if we change these dipoles to static toroidal (electric or magnetic) moments? The VP, obtained by putting $\omega = 0$ in (7.3), vanishes outside the treated spin chains (or tubes). This means that complete self-screening takes place. This property (self-screening) is conserved for the arbitrary continuous deformation of toroidal spin chains [13].

8. Conclusion

We briefly summarize the main results obtained:

(1) A realistic construction of the static electric solenoid is presented. Outside it there is an electric vector potential which cannot be eliminated by the gauge transformation. Concrete physical examples are given for which the electric vector potential, not the scalar one, has physical meaning.

(2) Non-static solenoids are considered. They emit waves of electromagnetic potentials propagating off the solenoid with the velocity of light. The conditions are discussed under which these waves can be detected.

We feel that the present paper raises more questions than answers. We rephrase the question posed by Aharonov and Bohm in their famous 1959 paper [32] in the following way: Do the electric vector potential of the static electric solenoid and the electromagnetic potentials of the non-static solenoid have the physical meaning? How can their existence be verified experimentally?

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